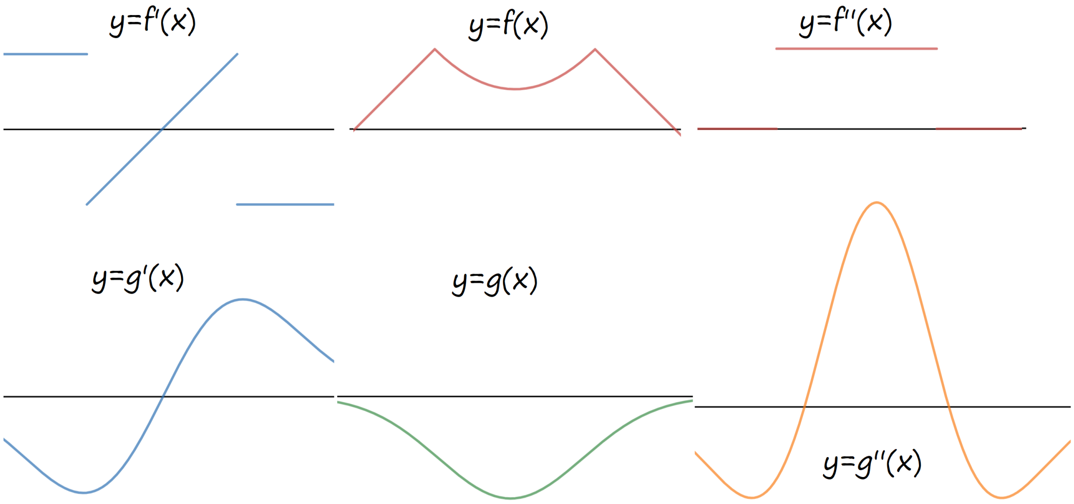
MATH 102:107, CLASS 11 (FRI SEPT 29)

(1) Sketch both the *derivative* and *antiderivative* of the following two graphs.

Solution:



(2) A ball is thrown with a vertical velocity of 10 m/s from a tower of height 50 m. Its height at time t is given by

$$h(t) = 50 + 10t - 5t^2$$

(a) At what time t does it reach its highest point?

Solution: We want the *maximum* of the function $h(t)$. This must necessarily happen when the derivative is zero (think local maximum), so we calculate $h'(t)$ and figure out where it is zero. $h'(t) = 10 - 10t$, and this equals zero precisely when $\boxed{t = 1}$. It is a local maximum because, at $t = 1$, $h'(t)$ is switching from positive to negative.

(b) How high is it at this highest point?

Solution: Plug $t = 1$ into $h(t)$ to get that the highest point is $h(1) = 50 + 10 - 5 = 55$.

Note: Another way to solve this problem, is by completing the square:

$$h(t) = 50 + 10t - 5t^2 = 55 - 5(t^2 - 2t + 1) = 55 - 5(t-1)^2$$

Since $-5(t-1)^2$ is always negative, $h(t)$ has a maximum possible value of 55, and this is achieved when $t = 1$.

(3) Let $f(x) = \frac{3}{x-2}$.

(a) Find the equation of the tangent line to $y = f(x)$ at the point $(3, 3)$.

Solution: We want the equation of the line which passes through $(3, 3)$ and has slope $f'(3)$. The point-slope formula tells us that this equation is

$$y - 3 = f'(3)(x - 3)$$

so we just need to calculate $f'(3)$. Since $f(x) = 3(x-2)^{-1}$, we apply the Chain Rule to get

$$f'(x) = -3(x-2)^{-2} = -\frac{3}{(x-2)^2} \implies f'(3) = -3$$

Thus, the tangent line at $(3, 3)$ has equation $y - 3 = -3(x - 3)$, which can also be written as $y = -3x + 12$.

(b) Find the equations of all tangent lines to $y = f(x)$ which are parallel to the line $y = -x$.

Solution: The line $y = -x$ has slope -1: therefore, we are looking for all values of x where $f'(x) = -1$. That is,

$$-\frac{3}{(x-2)^2} = -1 \iff 3 = (x-2)^2$$

$$\iff \pm\sqrt{3} = x - 2 \iff x = 2 \pm \sqrt{3}$$

. Let's find the equations of the tangent lines at these two values of x.

(i) At $x = 2 + \sqrt{3}$, $f(x) = \frac{3}{x-2} = \frac{3}{\sqrt{3}} = \sqrt{3}$. Thus, by point-slope, the equation of the line is $y - \sqrt{3} = -(x - (2 + \sqrt{3}))$, which can be rewritten as $\boxed{y = -x + 2 + 2\sqrt{3}}$.

(ii) At $x = 2 - \sqrt{3}$, $f(x) = \frac{3}{x-2} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$. Thus, by point-slope, the equation of the line is $y + \sqrt{3} = -(x - (2 - \sqrt{3}))$, which can be rewritten as $\boxed{y = -x + 2 - 2\sqrt{3}}$.

(c) (CHALLENGE) Find the equations of all tangent lines to $y = f(x)$ which pass through the point $(5, 3)$.

Solution: There are no such lines! If you look at the graph, $(5, 3)$ lies above and to the right of the graph of $y = f(x)$, and no tangent line can pass through it.

(4) Let $f(x) = x^3 - 3x + 1$.

(a) Find the equation of the tangent line at $x = 2$.

Solution: The equation of the tangent line is, by point-slope,

$$y - f(2) = f'(2)(x - 2)$$

so we must calculate $f(2)$ and $f'(2)$.

$$f(2) = 8 - 6 + 1 = 3$$

$$f'(x) = 3x^2 - 3 \implies f'(2) = 12 - 3 = 9$$

Thus the equation of the tangent line is $y - 3 = 9(x - 2)$, or $y = 9x - 15$.

(b) Where does this line intersect the x-axis?

Solution: We want to figure out where $9x - 15 = 0$: this happens when $x = 15/9 = 5/3$. So the tangent line intersects the x-axis at $(5/3, 0)$.

Note: You just performed one iteration of Newton's method for the function $f(x) = x^3 - 3x + 1$. We started with $x_0 = 2$, and found $x_1 = 5/3$. If you actually plug x_1 into the function,

$$f(5/3) = 125/27 - 3(5/3) + 1 = 125/27 - 108/27 = 17/27$$

so this is a better approximation to a root of f than $x_0 = 2$ was.

(5) In this question, we will approximate $\sqrt{105}$.

(a) Use linear approximation to approximate $\sqrt{105}$.

Solution: We use linear approximation on the function $f(x) = \sqrt{x}$, centered around $x = 100$. In other words, we will find the equation of the tangent line to $f(x)$ at $x = 100$, and then plug $x = 105$ into the equation of this tangent line. The tangent line has equation

$$y - f(100) = f'(100)(x - 100)$$

$$y - 10 = \frac{1}{20}(x - 100)$$

here $f'(100)$ was calculated by the fact that $f'(x) = \frac{1}{2\sqrt{x}}$.

$$y = 10 + \frac{1}{20}(x - 100)$$

Now plug in $x = 105$ to get $10 + \frac{1}{20}(5) = \frac{41}{4}$. Or, in decimal notation, 10.25.

(b) Use Newton's method to approximate $\sqrt{105}$. Find x_1 only. (Hint: What function $f(x)$ are you trying to find a root for?)

Solution: The number $\sqrt{105}$ is a root of the equation $f(x) = x^2 - 105$. Therefore, we use Newton's method to approximate a root of this equation. If we start with the guess $x_0 = 10$, then we calculate x_1 by drawing the tangent line at $(10, f(10))$, and then calculating its intersection with the x-axis. This tangent line has equation

$$y - f(10) = f'(10)(x - 10)$$

Using that $f(10) = -5$ and $f'(10) = 2(10) = 20$,

$$y + 5 = 20(x - 10)$$

To calculate the intersection with the x-axis, set $y = 0$:

$$5 = 20(x - 10) \iff \frac{1}{4} = x - 10 \iff x = \frac{41}{4}$$

Thus, $x_1 = \frac{41}{4}$, or 10.25 in decimal notation.

Note: One iteration of Newton's method gave us the same result as linear approximation. Yes, Newton's method is just iterated linear approximation!

(6) (Ladybugs and Aphids) In a garden, ladybugs (predator) and aphids (prey) both live. If x is the number of aphids, then the *reproduction rate* of the aphids, $G(x)$, and the *predation rate* by the ladybugs, are given by

$$G(x) = 3x \qquad\qquad P(x) = \frac{30x}{5 + x}$$

This behavior, when the predation rate is dependent entirely on x, is called a *functional response.* The particular function given here is an example of a Type II functional response.

(a) Is there a positive value of x where the reproduction rate and predation rate are equal? What happens when x is less than this value? Greater than this value?

Solution: We are asking whether there is some x such that $G(x) = P(x)$, i.e. where $3x = \frac{30x}{5+x}$. We can solve this equation.

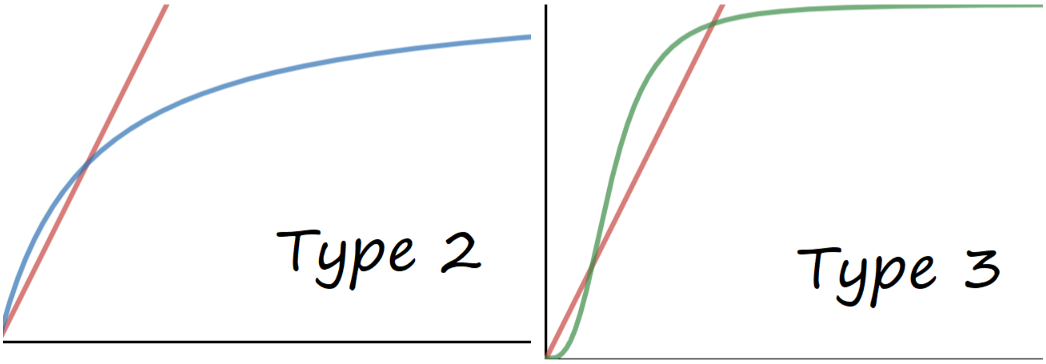
$$3x = \frac{30x}{5 + x} \iff 15x + 3x^2 = 30x \iff 3x(x - 5) = 0$$

which has solutions $x = 0, 5$. So the two rates balance each other out at $x = 5$.
For $0 < x < 5$, $P(x) > G(x)$, which leads to x decreasing further. In other words, the ladybugs eat the aphid population completely. For $x > 5$,

$P(x) < G(x)$, which leads to x increasing further. In other words, the aphids grow out of control. So this is an *unstable equilibrium.*

(b) Suppose now that the predation rate is instead given by $P(x) = \frac{30x^3}{50+x^3}$. Is there a positive value of x where $P(x) = G(x)$?

Solution: Here's a picture. The previous part of the problem was an example of a Type 2 functional response (left), and this part is Type 3 (right).



There are two values of x where $P(x) = G(x)$. One way to have figured this out without drawing the picture is to ask which positive values of x make $P(x) > G(x)$:

$$\frac{30x^3}{50+x^3} > 3x \iff 30x^3 > 3x(50+x^3) \iff 10x^2 > 50+x^3 \iff x^3 - 10x^2 + 50 < 0$$

and if you plug in $x = 1, 2, 3, 4, \ldots$ you will see that the function $f(x) = x^3 - 10x^2 + 50$ starts out positive, then becomes negative, and then switches back to being positive. This means that this function is zero at two points.

(c) We'll now use Newton's method to approximate one of these values of x. We want to find a value of x such that $P(x) - G(x) = 0$, i.e.

$$\frac{30x^3}{50+x^3} - 3x = 0 \iff 30x^3 - 3x(50+x^3) = 0 \iff x^3 - 10x^2 + 50 = 0$$

So finding x such that the predation rate equals the reproduction rate, is equivalent to finding roots of $f(x) = x^3 - 10x^2 + 50$. Starting with $x_0 = 3$, use one iteration of Newton's method to get x_1 which approximates a root of $f(x)$.

Solution: In the previous Newton's method problem, we went through the process of finding the equation of the tangent line at x_0, then set y equal to zero and solved for x to get x_1. That process, in general, will give you

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

You can check that this formula works with what we did in the last Newton's method problem. So we'll just use this formula this time.

$$x_1 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{27 - 90 + 50}{27 - 60} = 3 - \frac{17}{33} = \boxed{\frac{82}{33}}$$

In the picture above, this number is close to the first intersection (the one with smaller x).

(d) What happens to the ladybugs and aphids when x is *less* than this equilibrium value? *Greater* than this equilibrium value? What does this say about the dynamics of the two populations?

Solution: When x is less than this equilibrium point, $P(x) < G(x)$, and so the population of aphids grows. But when x is greater than this equilibrium point, $P(x) > G(x)$, and so the population of aphids shrinks. One says that this is a *stable equilibrium.*

(e) What if $P(x)$ were instead $\frac{30x^3}{500+x^3}$?

Solution: In this case (you can plot it on desmos) the two graphs do not intersect at all. The graph of $P(x)$ lies completely below the graph of $G(x)$. In this situation, the aphids grow out of control no matter what.